Probability Distributions and Simulations

Introduction

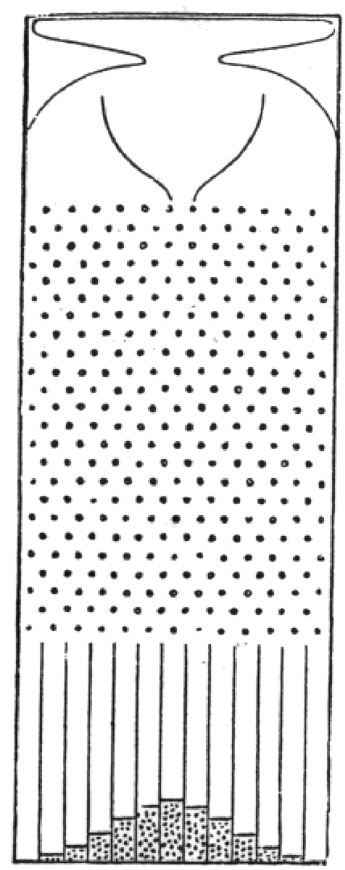
This activity combines a number of aspects of statistics.

- First you will use probability distributions to model the possible outcomes of a particular experiment, one with its roots in randomness.
- Second, you will use your model to estimate or, to a degree, predict the outcome of the experiment.
- Lastly, this activity will employ simulation to replicate the experiment allowing you to compare your estimates with an actual outcome.

The experiment is an old one coming from the days when probability was studied with the use of coins, marbles, pegs, cards, or whatever could be used. We will analyze the paths taken by marbles as they fall through vertical boards consisting of rows of pegs. This probability experiment is frequently simulated in Mathematics and Science Museums around the world.

Background

In the 1870s, Sir Francis Galton created a device he called a quincunx for studying probability. The device was made up of a vertical board with a chute at the top. The chute was filled with marbles which were dispensed into an array of pegs. The pegs acted as obstructions, forcing the marbles to change direction, the choice of direction being random. At the bottom of the quincunx was a set of bins for catching the marbles.



Galton's original sketch of the quincunx illustrates the setup and shows a possible arrangement of marbles in their bins after completing their journey. The idea being to study and explain that final distribution of marbles among bins.

Notice the arrangement of pegs as alternating rows so that between two pegs in one row there is a peg in the next row. This falling marble will strike a peg in each row as it progresses. In fact, the word quincunx refers to any arrangement of five objects in a rectangle, one object at each corner and one in the middle.

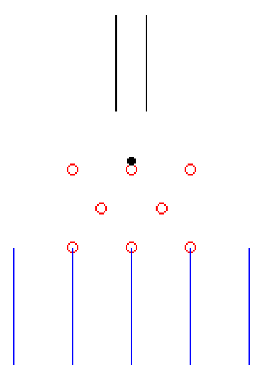
The word is often generalized to mean anything made up of such patterns of five. Our goal is to describe, mathematically and probabilistically, the possible resting places for a marble passing through the quincunx.

Mathematical Analysis

Click here: http://media.pearsoncmg.com/aw/aw_triola_elemstats_9/ip/quincunx1.gif to view an animation that simulates the path of one marble through a quincunx with 10 rows of pegs.

You can see the marble wend its way bouncing from peg to peg until it lands in the bin that it does. Note there are any number of ways the marble could have found its way to the same bin. Can you see another?

Let's look at a case with a small number of rows of pegs, say 3, with 4 bins at the bottom. The picture below shows the marble beginning its descent.



Number the bins 0 to 3 from left to right. Next assume that when a marble hits a peg, the probability is ½ the marble will drop to the left and ½ it will drop to the right.

How can the marble end up in the leftmost bin, bin number 0? The only way is if the marble drops to the left each time it strikes a peg. Since the probability is ½ at each stage, the probability of landing all the way to the left is

$(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = 1/8$

In fact, any specific path through the pegs will have probability 1/8.

Now, how can the marble land in bin number 1, the second bin from the left. There are three paths starting from the top which end in the second bin, namely:

bounce left, bounce left, bounce right
bounce left, bounce right, bounce left
bounce right, bounce left, bounce left

Each path has probability 1/8 so the probability of landing in the second bin is 3/8. Similarly (or by symmetry) the probability of landing in the third bin is 3/8 and the rightmost bin 1/8. We have completely determined the probabilities for the quincunx with 3 rows which we summarize in the table below.

Bin	0	1	2	3
Probability	1/8	3/8	3/8	1/8

Now that you understand the quincunx and how to compute the associated probabilities, you are ready to move on to the exercises for this activity.

Exercises

1. Suppose the bins at the bottom of the quincunx are numbered from 0 to N. Argue that the probability of a marble landing in bin number N follows a binomial distribution where N represents the number of trials and the probability p is ½.

2. Write down the formula that gives the probability for a marble to end up in the bin labeled X.

3. Complete a probability distribution table for the case N=8.

Bin	0	1	2	3	4	5	6	7	8
Probability									

4. If 350 marbles pass through the quincunx, how many marbles would you predict would land in each of the nine bins?

Bin	0	1	2	3	4	5	6	7	8
Probability									

5. Click here: http://media.pearsoncmg.com/aw/aw_triola_elemstats_9/ip/quincunx2.gif to view an animation that simulates the quincunx using 350 marbles and 8 rows of pegs. At the end of the video, count the number of marbles in each bin. Compare with your predictions from Exercise 3. Do they agree? Discuss why your predictions and the simulation may disagree.

6. Suppose you could build a quincunx that would accommodate any number of pegs, bins and marbles. Imagine the bins becoming narrower and the marbles shrinking to fit. What kind of distribution do you think you'd see as the number of bins increases and a near infinite number of marbles pass through?

As the rows fill up the marbles will begin to create a triangular shape getting more narrow at the top.